Chapter 1

Implicitni funkce

1.0.1 Postup a teorie

Necht $n, k \in \mathbb{N}$. V nasledujicich prikladech obdrzime jeden ci vice vztahu (rovnic $F(x_1, \dots, x_n) = \boldsymbol{o}$) a bod $A = [a_1, \dots, a_n]$ a budeme odpovidat na nektere z nasledujicich otazek:

(a) Za predpokladu, ze $i \in \{1, \dots, n\}$ a $F : \mathbb{R}^n \to \mathbb{R}$. Otazka zni, zda existuje funkce

$$x_i = f(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n),$$

ktera je definovana vztahem

$$H(\cdot) := F(x_1, \dots, x_{i-1}, f(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n), x_{i+1}, \dots, x_n) = 0$$

a pro kterou plati $f(a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_n) = a_i$. Pokud takova funkce existuje, tak se ptame, zda je tridy $\mathcal{C}^k$ na nejakem okoli bodu $[a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_n]$.

(b) Za predpokladu, ze $i < j \in \{1, \dots, n\}$ a $F = (F_1, F_2) : \mathbb{R}^n \to \mathbb{R}^2$. Otazka zni, zda existuji funkce

$$x_i = f_1(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_{j-1}, x_{j+1}, \dots, x_n),$$
$$x_j = f_2(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_{j-1}, x_{j+1}, \dots, x_n),$$

ktere jsou definovany vztahem

$$H(\cdot) := F(x_1, \dots, x_{i-1}, f_1(\cdot), x_{i+1}, \dots, x_{j-1}, f_2(\cdot), x_{j+1}, \dots, x_n) = \boldsymbol{o}$$

($H = (H_1, H_2)$) a pro ktere plati

$$f_1(a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_{j-1}, a_{j+1}, \dots, a_n) = a_i,$$
$$f_2(a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_{j-1}, a_{j+1}, \dots, a_n) = a_j.$$

Pokud takove funkce existuji, tak se ptame, zda jsou tridy $\mathcal{C}^k$ na nejakem okoli bodu $[a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_{j-1}, a_{j+1}, \dots, a_n]$.

(c) U nalezenych funkci dale zkoumame hodnotu prvni ci druhe derivace, tecnou rovinu, monotonii nebo konvexitu.

Na otazku (a) (respektive (b)) odpovidame za pomoci Vety o implicitni funkci (respektive Vety o implicitnich funkcich). Abychom mohli odpovedet pozitivne na tyto otazky, tak je treba overit nasledujici podminky:

(a) – $F(A) = 0$,

– Existuje okoli V bodu A, ze $F \in \mathcal{C}^k(V)$,

– $\frac{\partial F}{\partial x_i}(A) \neq 0$.

(b) – $F(A) = \boldsymbol{o}$,

– Existuje okoli V bodu A, ze $F \in \mathcal{C}^k(V)$ $(F_1, F_2 \in \mathcal{C}^k(V))$,

– $\begin{vmatrix} \frac{\partial F_1}{\partial x_i} & \frac{\partial F_1}{\partial x_j} \\ \frac{\partial F_2}{\partial x_i} & \frac{\partial F_2}{\partial x_j} \end{vmatrix}(A) \neq 0$.

(c) K vypoctu derivaci v pripade (a) derivujeme vztah

$$H(\cdot) = F(x_1, \dots, x_{i-1}, f(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n), x_{i+1}, \dots, x_n) = 0$$

a pouzivame retizkove pravidlo. Taktez budeme vyuzivat, ze pokud je funkce tridy $\mathcal{C}^2$, tak muzeme zamenovat poradi derivovani u derivaci druheho radu. Obdobne postupujeme v pripade (b). Odvodime si nasledujici 4 vztahy pomoci nichz budeme pocitat 1. a 2. derivace v pripade (a) a 1. derivace v pripade (b). Necht $s \neq v \in \{1, \dots, n\} \setminus \{i\}$ a $t \in \{1, \dots, n\} \setminus \{i, j\}$.

$$\begin{aligned} 0 =& \frac{\partial H}{\partial x_s} = \frac{\partial F}{\partial x_s} + \frac{\partial F}{\partial x_i} \cdot \frac{\partial f}{\partial x_s}, \\ & \frac{\partial f}{\partial x_s} = -\frac{\frac{\partial F}{\partial x_s}}{\frac{\partial F}{\partial x_i}}, \end{aligned} \tag{1.1}$$

$$\begin{aligned} 0 =& \frac{\partial^2 H}{\partial x_s^2} = \frac{\partial}{\partial x_s}\left(\frac{\partial F}{\partial x_s} + \frac{\partial F}{\partial x_i} \cdot \frac{\partial f}{\partial x_s}\right) \\ =& \frac{\partial^2 F}{\partial x_s^2} + \frac{\partial^2 F}{\partial x_s \partial x_i} \cdot \frac{\partial f}{\partial x_s} + \frac{\partial F}{\partial x_i} \cdot \frac{\partial^2 f}{\partial x_s^2} + \frac{\partial f}{\partial x_s}\left(\frac{\partial^2 F}{\partial x_i \partial x_s} + \frac{\partial^2 F}{\partial x_i^2} \cdot \frac{\partial f}{\partial x_s}\right), \\ & \frac{\partial^2 f}{\partial x_s^2} = -\frac{\frac{\partial^2 F}{\partial x_s^2} + \frac{\partial f}{\partial x_s}\left(2\frac{\partial^2 F}{\partial x_i \partial x_s} + \frac{\partial^2 F}{\partial x_i^2} \cdot \frac{\partial f}{\partial x_s}\right)}{\frac{\partial F}{\partial x_i}}, \end{aligned} \tag{1.2}$$

$$\begin{aligned} 0 =& \frac{\partial^2 H}{\partial x_s \partial x_v} = \frac{\partial}{\partial x_v}\left(\frac{\partial F}{\partial x_s} + \frac{\partial F}{\partial x_i} \cdot \frac{\partial f}{\partial x_s}\right) \\ =& \frac{\partial^2 F}{\partial x_s \partial x_v} + \frac{\partial^2 F}{\partial x_s \partial x_i} \cdot \frac{\partial f}{\partial x_v} + \frac{\partial F}{\partial x_i} \cdot \frac{\partial^2 f}{\partial x_s \partial x_v} + \frac{\partial f}{\partial x_s}\left(\frac{\partial^2 F}{\partial x_i \partial x_v} + \frac{\partial^2 F}{\partial x_i^2} \cdot \frac{\partial f}{\partial x_v}\right), \\ & \frac{\partial^2 f}{\partial x_s \partial x_v} = -\frac{\frac{\partial^2 F}{\partial x_s \partial x_v} + \frac{\partial^2 F}{\partial x_s \partial x_i} \cdot \frac{\partial f}{\partial x_v} + \frac{\partial f}{\partial x_s}\left(\frac{\partial^2 F}{\partial x_i \partial x_v} + \frac{\partial^2 F}{\partial x_i^2} \cdot \frac{\partial f}{\partial x_v}\right)}{\frac{\partial F}{\partial x_i}}, \end{aligned} \tag{1.3}$$

$$0 = \frac{\partial H_l}{\partial x_t} = \frac{\partial F_l}{\partial x_t} + \frac{\partial F_l}{\partial x_i} \cdot \frac{\partial f_1}{\partial x_t} + \frac{\partial F_l}{\partial x_j} \cdot \frac{\partial f_2}{\partial x_t}, \; l \in \{1, 2\}.$$

Z posledni rovnosti plyne, ze vektor $\left(\frac{\partial f_1}{\partial x_t}, \frac{\partial f_2}{\partial x_t}\right)$ je resenim soustavy linearnich rovnic s rozsirenou matici:

$$\left(\begin{array}{cc|c} \frac{\partial F_1}{\partial x_i} & \frac{\partial F_1}{\partial x_j} & -\frac{\partial F_1}{\partial x_t} \\ \frac{\partial F_2}{\partial x_i} & \frac{\partial F_2}{\partial x_j} & -\frac{\partial F_1}{\partial x_t} \end{array}\right). \tag{1.4}$$

Pokud je derivace v danem bode kladna, tak ze spojitosti derivace plyne, ze je kladna na nejakem okoli daneho bodu a tedy na okoli daneho bodu je funkce rostouci. Obdobne pro klesajici, konvexni nebo konkavni.

V pripade (a) tecna rovina splnuje nasledujici vztah:

$$\sum_{s=1}^{n} \frac{\partial F}{\partial x_s}(A)(x_s - a_s) = 0. \tag{1.5}$$

1.0.2 Priklady

V nasledujicich prikladech vzdy uvedu vztah (F), bod (A), jake promenne vyjadruji pomoci tohoto vztahu a co o nich chci dokazat. Matici druhych derivaci funkce F budeme znacit $\nabla^2 F$. K vypoctu reseni budeme pouzivat “Vypocty” a vyse odvozene vztahy.

Priklad 1.

- Zadani: $F(x, y) = x^2 + 2xy^2 + y^4 - y^5 = 0$, $A = [0, 1]$, $y = y(x)$.
- Otazky: $y(x) \in \mathcal{C}^\infty$? $y''(0) =$? Je $y(x)$ rostouci na nejakem okoli bodu 0?
- Vypocty:

$$\nabla F(x, y) = \left[2x + 2y^2, 4xy + 4y^3 - 5y^4\right], \qquad \nabla F(A) = [2, -1],$$
$$\nabla^2 F(x, y) = \begin{pmatrix} 2 & 4y \\ 4y & 4x + 12y^2 - 20y^3 \end{pmatrix}, \quad \nabla^2 F(A) = \begin{pmatrix} 2 & 4 \\ 4 & -8 \end{pmatrix}.$$

- Reseni: Protoze $F \in \mathcal{C}^\infty(\mathbb{R}^2)$, $F(A) = 0$ a $\frac{\partial F}{\partial y}(A) = -1 \neq 0$, tak z vety o IF plyne, ze $y \in \mathcal{C}^\infty$ na nejakem okoli bodu 0. Z (1.1) a (1.2) plyne, ze $y''(0) = -14$ ($y'(0) = 2$). Ano, je rostouci.

Priklad 2.

- Zadani: $F(x, y) = x^3 - x^2 y + y^2 - xy - 1 = 0$, $A = [0, 1]$, $y = y(x)$.
- Otazky: $y(x) \in \mathcal{C}^\infty$? $y''(0) =$? Je $y(x)$ rostouci na nejakem okoli bodu 0?
- Vypocty:

$$\nabla F(x, y) = \left[3x^2 - 2xy - y, -x^2 + 2y - x\right], \qquad \nabla F(A) = [-1, 2],$$
$$\nabla^2 F(x, y) = \begin{pmatrix} 6x - 2y & -2x - 1 \\ -2x - 1 & 2 \end{pmatrix}, \qquad \nabla^2 F(A) = \begin{pmatrix} -2 & -1 \\ -1 & 2 \end{pmatrix}.$$

- Reseni: Protoze $F \in \mathcal{C}^\infty(\mathbb{R}^2)$, $F(A) = 0$ a $\frac{\partial F}{\partial y}(A) = 2 \neq 0$, tak z vety o IF plyne, ze $y \in \mathcal{C}^\infty$ na nejakem okoli bodu 0. Z (1.1) a (1.2) plyne, ze $y''(0) = \frac{5}{4}$ ($y'(0) = \frac{1}{2}$). Ano, je rostouci.

Priklad 3.

- Zadani: $x^3 + y^3 = \log\left(\frac{x^2 + y^2}{2}\right)$, $A = [1, -1]$, $x = x(y)$.
- Otazky: $x(y) \in \mathcal{C}^\infty$? $x''(-1) =$? Je $x(y)$ konkavni na nejakem okoli bodu -1?

- Vypocty:

$$F(x,y) := x^3 + y^3 - \log\left(\frac{x^2+y^2}{2}\right),$$

$$\nabla F(x,y) = \left[3x^2 - \frac{2x}{x^2+y^2}, 3y^2 - \frac{2y}{x^2+y^2}\right], \qquad \nabla F(A) = [2,4],$$

$$\nabla^2 F(x,y) = \begin{pmatrix} 6x + 2\frac{x^2-y^2}{(x^2+y^2)^2} & \frac{4xy}{(x^2+y^2)^2} \\ \frac{4xy}{(x^2+y^2)^2} & 6y - 2\frac{x^2-y^2}{(x^2+y^2)^2} \end{pmatrix},$$

$$\nabla^2 F(A) = \begin{pmatrix} 6 & -1 \\ -1 & -6 \end{pmatrix}.$$

- Reseni: Protoze $F \in \mathcal{C}^\infty(\mathbb{R}^2 \setminus \{[0,0]\})$, $F(A) = 0$ a $\frac{\partial F}{\partial x}(A) = 2 \neq 0$, tak z vety o IF plyne, ze $x \in \mathcal{C}^\infty$ na nejakem okoli bodu -1. Z (1.1) a (1.2) plyne, ze $x''(-1) = -11$ ($x'(-1) = -2$). Ano, je konkavni.

Priklad 4.

- Zadani: $e^{2x+7y} = \log(1+x^2+y^2) + 1 + y$, $A = [0,0]$, $y = y(x)$, $x = x(y)$.
- Otazky: $y(x), x(y) \in \mathcal{C}^\infty$? $y''(0) =$? Tecna ke grafu funkce $x(y)$ v bode $[A]$?
- Vypocty:

$$F(x,y) := e^{2x+7y} - \log(1+x^2+y^2) - y - 1,$$

$$\nabla F(x,y) = \left[2e^{2x+7y} - \frac{2x}{1+x^2+y^2}, 7e^{2x+7y} - \frac{2y}{1+x^2+y^2} - 1\right],$$

$$\nabla F(A) = [2,6],$$

$$\nabla^2 F(x,y) = \begin{pmatrix} 4e^{2x+7y} - \frac{2(1-x^2+y^2)}{(1+x^2+y^2)^2} & 14e^{2x+7y} + \frac{4xy}{(1+x^2+y^2)^2} \\ 14e^{2x+7y} + \frac{4xy}{(1+x^2+y^2)^2} & 49e^{2x+7y} - \frac{2(1+x^2-y^2)}{(1+x^2+y^2)^2} \end{pmatrix},$$

$$\nabla^2 F(A) = \begin{pmatrix} 2 & 14 \\ 14 & 47 \end{pmatrix}.$$

- Reseni: Protoze $F \in \mathcal{C}^\infty(\mathbb{R}^2)$, $F(A) = 0$, $\frac{\partial F}{\partial y}(A) = 6 \neq 0$ a $\frac{\partial F}{\partial x}(A) = 2 \neq 0$, tak z vety o IF plyne, ze $x, y \in \mathcal{C}^\infty$ na nejakem okoli bodu 0. Z (1.1) a (1.2) plyne, ze $y''(0) = \frac{19}{54}$ ($y'(0) = -\frac{1}{3}$). Z (1.1) nebo (1.5) plyne, ze $T(y) = -3y$ ($x'(0) = -3$).

Priklad 5.

- Zadani: $F(x,y) = \log(x^2 + y^2 + \cos(xy)) + y = 0$, $A = [0,0]$, $y = y(x)$.
- Otazky: $y(x) \in \mathcal{C}^\infty$? $y''(0) =$? Nabyva funkce y v bode 0 lokalniho maxima?

- Vypocty:

$$\nabla F(x,y) = \left[\frac{2x - y\sin(xy)}{x^2+y^2+\cos(xy)}, 1 + \frac{2y - x\sin(xy)}{x^2+y^2+\cos(xy)}\right], \ \nabla F(A) = [0,1],$$

$$\frac{\partial^2 F}{\partial x^2}(x,y) = \frac{(2 - y^2\cos(xy))(x^2+y^2+\cos(xy)) - (2x - y\sin(xy))^2}{(x^2+y^2+\cos(xy))^2},$$

$$\frac{\partial^2 F}{\partial x \partial y}(x,y) = \frac{(-\sin(xy) - xy\cos(xy))(x^2+y^2+\cos(xy))}{(x^2+y^2+\cos(xy))^2} - \frac{(2x - y\sin(xy))(2y - x\sin(xy))}{(x^2+y^2+\cos(xy))^2},$$

$$\frac{\partial^2 F}{\partial y \partial x}(x,y) = \frac{\partial^2 F}{\partial x \partial y}(x,y), \ \frac{\partial^2 F}{\partial y^2}(x,y) = \frac{\partial^2 F}{\partial x^2}(y,x),$$

$$\nabla^2 F(A) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}.$$

- Reseni: Protoze $F \in \mathcal{C}^\infty(\mathbb{R}^2)$, $F(A) = 0$ a $\frac{\partial F}{\partial y}(A) = 1 \neq 0$, tak z vety o IF plyne, ze $y \in \mathcal{C}^\infty$ na nejakem okoli bodu 0. Z (1.1) a (1.2) plyne, ze $y''(0) = -2$ ($y'(0) = 0$). Ano, nabyva, neb derivace je nulova a funkce je konkavni na nejakem okoli bodu 0. Podrobnejsi reseni viz Zeleny IF9

Priklad 6.

- Zadani: $e^{xy} + xyz = e + 1$, $A = [1,1,1]$, $x = x(y,z)$, $y = y(x,z)$.

- Otazky: $x(y,z), y(x,z) \in \mathcal{C}^\infty$? $\frac{\partial^2 y}{\partial x \partial z}(1,1) =$? Tecna ke grafu funkce $x(y,z)$ v bode A?

- Vypocty:

$$F(x,y,z) := e^{xy} + xyz - e - 1,$$

$$\nabla F(x,y,z) = [ye^{xy} + yz, xe^{xy} + xy, xy], \ \nabla F(A) = [e+1, e+1, 1],$$

$$\nabla^2 F(x,y,z) = \begin{pmatrix} y^2 e^{xy} & (1+xy)e^{xy} + z & y \\ (1+xy)e^{xy} + z & x^2 e^{xy} & x \\ y & x & 0 \end{pmatrix},$$

$$\nabla^2 F(A) = \begin{pmatrix} e & 2e+1 & 1 \\ 2e+1 & e & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

- Reseni: Protoze $F \in \mathcal{C}^\infty(\mathbb{R}^3)$, $F(A) = 0$, $\frac{\partial F}{\partial x}(A) = \frac{\partial F}{\partial y}(A) = e + 1 \neq 0$, tak z vety o IF plyne, ze $x, y \in \mathcal{C}^\infty$ na nejakem okoli bodu $[1,1]$. Z (1.1) a (1.3) plyne, ze $\frac{\partial^2 y}{\partial x \partial z}(1,1) = \frac{1}{e+1}$ ($\frac{\partial y}{\partial x}(1,1) = -1$). Z (1.5) plyne, ze $T(y,z) = -y + 2 - \frac{z-1}{e+1}$. Podrobnejsi reseni viz. implicitka anglicky/IF 3.

Priklad 7.

- Zadani: $\sin(x-y) + x^2y^3z^4 = 1$, $A = [1,1,1]$, $x = x(y,z)$, $y = y(x,z)$, $z = z(x,y)$.

- Otazky: $x(y,z), y(x,z), z(x,y) \in \mathcal{C}^\infty$? $\frac{\partial^2 x}{\partial z^2}(1,1) =$? $\frac{\partial^2 z}{\partial x \partial y}(1,1) =$? Tecna rovina ke grafu funkce $y(x,z)$ v bode A?
- Vypocty:

$$\begin{aligned} F(x,y,z) :=& \sin(x-y) + x^2y^3z^4 - 1, \\ \nabla F(x,y,z) =& \left[\cos(x-y) + 2xy^3z^4, -\cos(x-y) + 3x^2y^2z^4, 4x^2y^3z^3\right], \\ \nabla F(A) =& [3,2,4], \\ \nabla^2 F(x,y,z) =& \begin{pmatrix} -\sin(x-y) + 2y^3z^4 & \sin(x-y) + 6xy^2z^4 & 8xy^3z^3 \\ \sin(x-y) + 6xy^2z^4 & -\sin(x-y) + 6x^2yz^4 & 12x^2y^2z^3 \\ 8xy^3z^3 & 12x^2y^2z^3 & 12x^2y^3z^2 \end{pmatrix}, \\ \nabla^2 F(A) =& \begin{pmatrix} 2 & 6 & 8 \\ 6 & 6 & 12 \\ 8 & 12 & 12 \end{pmatrix}. \end{aligned}$$

- Reseni: Protoze $F \in \mathcal{C}^\infty(\mathbb{R}^3)$, $F(A) = 0$, $\frac{\partial F}{\partial x}(A) = 3 \neq 0$, $\frac{\partial F}{\partial y}(A) = 2 \neq 0$ a $\frac{\partial F}{\partial z}(A) = 4 \neq 0$, tak z vety o IF plyne, ze $x, y, z \in \mathcal{C}^\infty$ na nejakem okoli bodu $[1,1]$. Z (1.1) a (1.2) plyne, ze $\frac{\partial^2 x}{\partial z^2}(1,1) = \frac{52}{27}$ ($\frac{\partial x}{\partial z}(1,1) = -\frac{4}{3}$). Z (1.1) a (1.3) plyne, ze $\frac{\partial^2 z}{\partial x \partial y}(1,1) = \frac{5}{8}$ ($\nabla z(1,1) = -\frac{1}{4}[3,2]$). Z (1.5) plyne, ze $T(x,z) = -\frac{3}{2}x - 2z + \frac{9}{2}$.

Priklad 8.

- Zadani: $F(x,y,z) = 2x^3y - 3x^2z^3 + 5zy^2 - 4x + 5y - z - 4 = 0$, $A = [1,1,1]$, $z(x,y)$.
- Otazky: $z(x,y) \in \mathcal{C}^\infty$? $\nabla z(1,1) =$? $\nabla^2 z(1,1) =$? Tecna ke grafu funkce $z(x,y)$ v bode A?
- Vypocty:

$$\begin{aligned} \nabla F(x,y,z) =& \left[6x^2y - 6xz^3 - 4, 2x^3 + 10zy + 5, -9x^2z^2 + 5y^2 - 1\right], \\ \nabla F(A) =& [-4, 17, -5], \\ \nabla^2 F(x,y,z) =& \begin{pmatrix} 12xy - 6z^3 & 6x^2 & -18xz^2 \\ 6x^2 & 10z & 10y \\ -18xz^2 & 10y & -18x^2z \end{pmatrix}, \\ \nabla^2 F(A) =& \begin{pmatrix} 6 & 6 & -18 \\ 6 & 10 & 10 \\ -18 & 10 & -18 \end{pmatrix}. \end{aligned}$$

- Reseni: Protoze $F \in \mathcal{C}^\infty(\mathbb{R}^3)$, $F(A) = 0$ a $\frac{\partial F}{\partial z}(A) = -5 \neq 0$, tak z vety o IF plyne, ze $z \in \mathcal{C}^\infty$ na nejakem okoli bodu $[1,1]$. Z (1.1) plyne, ze $\nabla z(1,1) = \frac{1}{5}[-4, 17]$. Z (1.2) a (1.3) plyne, ze $\nabla^2 z(1,1) = \frac{1}{125}\begin{pmatrix} 582 & -356 \\ -356 & -3252 \end{pmatrix}$. Z (1.5) plyne, ze $T(x,y) = -\frac{4}{5}(x-1) + \frac{17}{5}(y-1) + 1$.

Priklad 9.

- Zadani: $F(x,y,z,w) = 2xy^2zw - 3x^4w + 5yw^2x - 3w + 2zy - 3x = 0$, $A = [1,1,1,1]$, $z(x,y,w)$, $y(x,z,w)$.
- Otazky: $y(x,z,w), z(x,y,w) \in \mathcal{C}^\infty$? $\nabla z(1,1,1) =$? $\frac{\partial^2 z}{\partial w \partial x} =$? Tecna ke grafu funkce $y(x,z,w)$ v bode A?
- Vypocty:

$$\begin{aligned}\nabla F(x,y,z,w) =& \big[2y^2zw - 12x^3w + 5yw^2 - 3, 4xyzw + 5w^2x + 2z, 2xy^2w + 2y, \\ & 2xy^2z - 3x^4 + 10xyw - 3\big], \\ \nabla F(A) =& [-8, 11, 4, 6], \\ \nabla^2 F(x,y,z,w) =& \begin{pmatrix} -36x^2w & 4yzw + 5w^2 & 2y^2w & 2y^2z - 12x^3 + 10yw \\ 4yzw + 5w^2 & 4xzw & 4xyw + 2 & 4xyz + 10xw \\ 2y^2w & 4xyw + 2 & 0 & 2xy^2 \\ 2y^2z - 12x^3 + 10yw & 4xyz + 10xw & 2xy^2 & 10xy \end{pmatrix}, \\ \nabla^2 F(A) =& \begin{pmatrix} -36 & 9 & 2 & 0 \\ 9 & 4 & 6 & 14 \\ 2 & 6 & 0 & 2 \\ 0 & 14 & 2 & 10 \end{pmatrix}.\end{aligned}$$

- Reseni: Protoze $F \in \mathcal{C}^\infty(\mathbb{R}^4)$, $F(A) = 0$, $\frac{\partial F}{\partial y}(A) = 11 \neq 0$ a $\frac{\partial F}{\partial z}(A) = 4 \neq 0$, tak z vety o IF plyne, ze $y, z \in \mathcal{C}^\infty$ na nejakem okoli bodu $[1,1,1]$. Z (1.1) plyne, ze $\nabla z(1,1,1) = \frac{1}{4}[8, -11, -6]$. Z (1.3) plyne, ze $\frac{\partial^2 z}{\partial w \partial x}(1,1,1) = -\frac{1}{4}$. Z (1.5) plyne, ze $T(x,z,w) = \frac{1}{11}(8x - 4z - 6w + 13)$.

Priklad 10.

- Zadani: $x^2 + 2y^2 + 3z^2 + 4w^2 = 10xyzw$, $A = [1,1,1,1]$, $y = f(x,z,w)$,

$$T(x,z,w) := f(f^2(x,z,w), xzw, zf(x,z,w)).$$

- Otazky: $f, T \in \mathcal{C}^\infty$? $\frac{\partial T}{\partial z}(1,1,1) =$?
- Vypocty:

$$\begin{aligned} F(x,y,z,w) :=& x^2 + 2y^2 + 3z^2 + 4w^2 - 10xyzw, \\ \nabla F(x,y,z,w) =& \big[2x - 10yzw, 4y - 10xzw, 6z - 10xyw, 8w - 10xyz\big], \\ \nabla F(A) =& [-8, -6, -4, -2], \\ \nabla f(1,1,1) =& -\frac{1}{3}[4, 2, 1], \\ \frac{\partial T}{\partial z}(x,y,z) =& \frac{\partial f}{\partial x}(\cdot) 2f(\cdot) \frac{\partial f}{\partial z}(\cdot) + \frac{\partial f}{\partial z}(\cdot) xw + \frac{\partial f}{\partial w}(\cdot) \left(f(\cdot) + z \frac{\partial f}{\partial z}(\cdot) \right). \end{aligned}$$

- Reseni: Protoze $F \in \mathcal{C}^\infty(\mathbb{R}^4)$, $F(A) = 0$ a $\frac{\partial F}{\partial y}(A) = -6 \neq 0$, tak z vety o IF plyne, ze $f \in \mathcal{C}^\infty \Rightarrow T \in \mathcal{C}^\infty$ na nejakem okoli bodu $[1,1,1]$. $\frac{\partial T}{\partial z}(1,1,1) = 1$. Vyuzivame, ze $f^2(x,z,w) = xzw = zf(x,z,w) = 1$ v bode $[1,1,1]$. Podobny priklad viz. implicitka anglicky/IF 1.

Priklad 11.

- Zadani:

$$\begin{aligned} xe^u + v + 2uv =& 1, \quad A = [1,2,0,0], \\ ye^{u-v} - \frac{u}{1+v} =& 2x, \quad u = u(x,y) \quad v = v(x,y). \end{aligned}$$

- Otazky: $u(x,y), v(x,y) \in \mathcal{C}^\infty$? $\nabla u(1,2) =$? $\nabla v(1,2) =$? Tecna ke grafu funkce $v(x,y)$ v bode $[1,2,0]$?

- Vypocty:

$$\begin{aligned} F_1(x,y,z,w) :=& xe^u + v + 2uv - 1, \; F_2(x,y,z,w) := ye^{u-v} - \frac{u}{1+v} - 2x, \\ \nabla F_1(x,y,z,w) =& \left[e^u, 0, xe^u + 2v, 1 + 2u\right], \\ \nabla F_2(x,y,z,w) =& \left[-2, e^{u-v}, ye^{u-v} - \frac{1}{1+v}, -ye^{u-v} + \frac{u}{(1+v)^2}\right], \\ \nabla F_1(A) =& [1,0,1,1], \; \nabla F_2(A) = [-2,1,1,-2]. \end{aligned}$$

- Reseni: Protoze $F_1, F_2 \in \mathcal{C}^\infty(B(A,1))$ $(\rho_e(A,[x,y,u,-1]) \geq 1)$, $F_1(A) = F_2(A) = 0$ a $\begin{vmatrix} 1 & 1 \\ 1 & -2 \end{vmatrix} = -3 \neq 0$, tak z vety o IF plyne, ze $u, v \in \mathcal{C}^\infty$ na nejakem okoli bodu $[1,2]$. Z (1.4) plyne, ze $\nabla u(1,1)$ a $\nabla v(1,1)$ obdrzime vyresenim nasledujicich soustav linearnich rovnic:

$$\left(\begin{array}{cc|c|c} 1 & 1 & -1 & 0 \\ 1 & -2 & 2 & -1 \end{array}\right) \sim \left(\begin{array}{cc|c|c} 1 & 0 & 0 & -\frac{1}{3} \\ 0 & 1 & -1 & \frac{1}{3} \end{array}\right)$$

Tedy $\nabla u(1,2) = \left[0, -\frac{1}{3}\right]$ a $\nabla v(1,2) = \left[-1, \frac{1}{3}\right]$. $T(x,y) = -(x-1) + \frac{1}{3}(y-2)$. Podrobnejsi reseni casti prikladu viz. implicitka anglicky/IF 2.

Priklad 12.

- Zadani:

$$\begin{aligned} 6xyz - x - 2y - 3z =& 5, \quad A = [0,-1,-1], \\ e^{xy} =& yz, \quad y = y(x) \quad z = z(x). \end{aligned}$$

- Otazky: $y(x), z(x) \in \mathcal{C}^\infty$? $y''(0) =$? $z''(0) =$? Je $y(x)$ konkavni na nejakem okoli bodu 0?

- Vypocty:

$$\begin{aligned} F_1(x,y,z) :=& 6xyz - x - 2y - 3z - 5, & F_2(x,y,z) :=& e^{xy} - yz, \\ \nabla F_1(x,y,z) =& \left[6yz - 1, 6xz - 2, 6xy - 3\right], & \nabla F_1(A) =& \left[5,-2,-3\right], \\ \nabla F_2(x,y,z) =& \left[ye^{xy}, xe^{xy} - z, -y\right], & \nabla F_2(A) =& \left[-1,1,1\right], \end{aligned}$$

$$\nabla^2 F_1(x,y,z) = \begin{pmatrix} 0 & 6z & 6y \\ 6z & 0 & 6x \\ 6y & 6x & 0 \end{pmatrix}, \quad \nabla^2 F_1(A) = \begin{pmatrix} 0 & -6 & -6 \\ -6 & 0 & 0 \\ -6 & 0 & 0 \end{pmatrix},$$

$$\nabla^2 F_2(x,y,z) = \begin{pmatrix} y^2 e^{xy} & (1+xy)e^{xy} & 0 \\ (1+xy)e^{xy} & x^2 e^{xy} & -1 \\ 0 & -1 & 0 \end{pmatrix},$$

$$\nabla^2 F_2(A) = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}.$$

- Reseni: Protoze $F_1, F_2 \in \mathcal{C}^\infty(\mathbb{R}^3)$, $F_1(A) = F_2(A) = 0$ a $\begin{vmatrix} -2 & -3 \\ 1 & 1 \end{vmatrix} = 1 \neq 0$, tak z vety o IF plyne, ze $y, z \in \mathcal{C}^\infty$ na nejakem okoli bodu 0. Pro $i \in \{1,2\}$ spocitame

$$\begin{aligned} 0 = (F_i(x,y(x),z(x)))'' =& \frac{\partial^2 F_i}{\partial x^2}(\cdot) + 2\frac{\partial^2 F_i}{\partial x \partial y}(\cdot)y'(x) + 2\frac{\partial^2 F_i}{\partial x \partial z}(\cdot)z'(x) \\ &+ 2\frac{\partial^2 F_i}{\partial z \partial y}(\cdot)y'(x)z'(x) + \frac{\partial^2 F_i}{\partial y^2}(y'(x))^2 + \frac{\partial^2 F_i}{\partial z^2}(z'(x))^2 \\ &+ \frac{\partial F_i}{\partial y}(\cdot)y''(x) + \frac{\partial F_i}{\partial z}(\cdot)z''(x). \end{aligned} \tag{1.6}$$

Z (1.4) obdrzime system linearnich rovnic (I), jehoz vyresenim spocteme $y'(0) = -2$, $z'(0) = 3$. Nasledne dosazenim do (1.6) obdrzime druhy system linearnich rovnic (II), jehoz vyresenim spocitame $y''(0) = -15$ a $z''(0) = 6$. Tedy $y(x)$ je konkavni na nejakem okoli bodu 0.

$$(I): \quad \left(\begin{array}{cc|c} -2 & -3 & -5 \\ 1 & 1 & 1 \end{array}\right) \sim \left(\begin{array}{cc|c} 0 & 1 & 3 \\ 1 & 0 & -2 \end{array}\right),$$

$$(II): \quad \left(\begin{array}{cc|c} -2 & -3 & 12 \\ 1 & 1 & -9 \end{array}\right) \sim \left(\begin{array}{cc|c} 0 & 1 & 6 \\ 1 & 0 & -15 \end{array}\right).$$